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Anyon Basis of $c = 1$ Conformal Field Theory

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Abstract

We study the $c = 1$ conformal field theory of a free compactified boson with radius $r = \sqrt{\beta}$ (β is an integer). The Fock space of this boson is constructed in terms of anyon vertex operators and each state is labeled by an infinite set of pseudo-momenta of filled particles in pseudo-Dirac sea. Wave function of multi anyon state is described by an eigenfunction of the Calogero-Sutherland (CS) model. The $c = 1$ conformal field theory at $r = \sqrt{\beta}$ gives a field theory of CS model. This is a natural generalization of the boson-fermion correspondence in one dimension to boson-anyon correspondence. There is also an interesting duality between anyon with statistics $\theta = \pi/\beta$ and particle with statistics $\theta = \beta\pi$.

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1 Introduction

Recently there is a renewal of interest in the one-dimensional (1d) solvable model with inverse-square type interactions. This class of models includes Calogero-Sutherland model [1, 2], spin 1/2 generalization by Haldane and Shastry [3, 4], models with internal symmetry [5, 6, 7, 8], supersymmetric t-J model [9] or hierarchical generalization [10]. Haldane-Shastry model has been most deeply analyzed and it is understood that the elementary excitations are described by spinons with semionic statistics [11]. The ground state of the model is called Gutzwiller wave function, a good variational wave function for the Heisenberg model. Different from the Heisenberg model, the correlation function of the Haldane-Shastry model has no logarithmic correction and it indicates absence of marginally irrelevant operators in the theory. This means that the Haldane-Shastry model is the fixed point Hamiltonian of the Heisenberg model [12], which is the Wess-Zumino-Witten model of $SU(2)$ at level 1. This was confirmed recently [13, 14]. Since other models with inverse-square interactions also have no logarithmic corrections in their correlation functions, it is natural to think that all the models of this type are the fixed point Hamiltonian of interacting theories. In particular, there are many indications [15] that the Calogero-Sutherland (CS) model with coupling constant β on a circle

$$H_{cs} = \sum_i \frac{p_i^2}{2} + \left(\frac{\pi}{L}\right)^2 \sum_{i < j} \frac{\beta^2 - \beta}{\sin^2 \frac{\pi}{L}(x_i - x_j)}. \quad (1.1)$$

will give the $c = 1$ conformal field theory of a compactified boson with radius $r = \sqrt{\beta}$.

The $c = 1$ conformal field theory (CFT) of a compactified boson has a rich structure [16]. At the special radius point $r = 1$, the bosonic Fock space is also described by free Dirac fermions. At $r = \sqrt{2}$, there is an affine $SU(2)$ symmetry and Fock space is constructed by these generators. At $r = \sqrt{3}$, there is an $N = 2$ supersymmetry. Among them, $r = 1$ theory is special because, in fermionic representation, each state is labeled by an infinite set of momenta of filled particles in the Dirac sea. For example, if we mode-expand fermion operator $\psi^\dagger(z)$ by

$$\psi^\dagger(z) = \sum_{n \in \mathbf{Z}} \psi_{-n-1/2}^\dagger z^n, \quad (1.2)$$

the charge 0 ground state is written as

$$|0\rangle = \psi_{1/2}^\dagger \psi_{3/2}^\dagger \cdots |-\infty\rangle = \prod_{i=0}^{\infty} \psi_{-m_i^0}^\dagger |-\infty\rangle \quad (1.3)$$

where $m_i^0 = -(2i-1)/2$. In general, a state in the fermion basis is written as

$$|\{m_i\}\rangle = \prod_{i=0}^{\infty} \psi_{-m_i}^\dagger |-\infty\rangle. \quad (1.4)$$

$\{m_i\}$ ($m_i > m_j$ if $i < j$) are infinite set of momenta of filled particles in the Dirac sea. No such structure is known for other compactified bosons.

In this paper, we show that, as well as the free fermion theory, there is a pseudo-momentum representation of states at other radius points $r = \sqrt{\beta}$, where β is an integer. At these points, the Fock space can be created by anyon vertex operators. The pseudo-Dirac sea is constructed by fermion or boson vertex operators corresponding to odd β or even β . Each state is labeled by an infinite set of integers m_i ($i = 1, 2, \dots$) where $m_1 > m_2 > \dots$ but, different from the free fermion, there is a restriction $|m_i - m_j| \geq \beta$ to the value of pseudo-momenta. We mainly consider a case with integer β but it is straightforward to generalize the result to a case with rational $\beta = p/q$.

The paper is organized as follows. In section 2, we give a brief review of Jack symmetric functions and the Calogero-Sutherland model. In section 3, we first construct $c = 1$ CFT Fock space by multi-anyons (Section 3.3) and then by *super fermion* vertex operators (Sec.3.4). By using these construction, we give a pseudo-Dirac sea description of the Fock space (Sec.3.5). We also give a collective Hamiltonian (Sec.3.6). Section 4 is devoted to conclusion and discussions.

2 Symmetric Functions and Calogero-Sutherland Model

2.1 Jack Polynomials

First we give a brief review of Jack symmetric functions. Here we use the notation of Macdonald [17]. Let (x_1, \dots, x_N) be independent indeterminates.

There are many basis of symmetric functions for these valuables, each of which are characterized by a *partition*, or Young tableau. It is a sequence

$$\{\lambda\} = (\lambda_1, \lambda_2, \dots) \quad (2.1)$$

of non-negative integers, such that $\lambda_1 \geq \lambda_2 \geq \dots$. The degree of the partition is defined by

$$|\lambda| = \sum \lambda_i. \quad (2.2)$$

The conjugate partition $\{\lambda'\}$ is defined to interchange rows and columns in the corresponding Young tableau. The nonzero λ_i are called the parts of the partition and the number of the parts is the length $l(\lambda)$ of the partition. If $\{\lambda\}$ has m_1 parts equal to 1, m_2 parts equal to 2, and so on, the partition is written as $\{\lambda\} = (1^{m_1} 2^{m_2} \dots)$. To this partition, we define

$$z_\lambda \equiv \prod_r r^{m_r} m_r! \quad (2.3)$$

The *partial ordering* is defined in the set of all partitions as follows:

$$\{\lambda\} \geq \{\mu\} \iff |\lambda| = |\mu| \quad \text{and} \quad \sum_i^r \lambda_i \geq \sum_i^r \mu_i \quad \text{for all } r \geq 1. \quad (2.4)$$

For each partition we define the following basis of symmetric functions;

(1) Monomial symmetric functions

To each partition $\{\lambda\}$ monomials $x_{\{\lambda\}}$ are defined by $x_{\{\lambda\}} = x_1^{\lambda_1} x_2^{\lambda_2} \dots$. Monomial symmetric functions $m_{\{\lambda\}}$ are obtained from $x_{\{\lambda\}}$ by permutations of the x 's.

(2) Power sums

N -th power sum is defined by $p_n = \sum_i x_i^n$. For each partition, $p_{\{\lambda\}}$ is defined by $p_{\{\lambda\}} = p_1^{\lambda_1} p_2^{\lambda_2} \dots$.

Now we define the Jack symmetric functions. Let's define a scalar product² on a space of symmetric functions

$$\langle p_{\{\lambda\}}, p_{\{\mu\}} \rangle = \delta_{\{\lambda\}, \{\mu\}} \beta^{-l(\lambda)} z_\lambda. \quad (2.5)$$

²Note that α in the Macdonald's notation [17] or in the Stanley's notation [18] is $\alpha = 1/\beta$.

Then one can define the Jack symmetric functions $J_{\{\lambda\}}^\beta(x_1, \dots, x_N)$ by the following conditions:

$$\begin{aligned} (a) \quad & J_{\{\lambda\}}^\beta(x_i) = m_{\{\lambda\}} + \sum_{\{\mu\} < \{\lambda\}} v_{\{\lambda\}, \{\mu\}}(\beta) m_{\{\mu\}} \\ (b) \quad & \langle J_{\{\lambda\}}^\beta, J_{\{\mu\}}^\beta \rangle = 0, \quad \text{if } \{\lambda\} \neq \{\mu\}. \end{aligned} \quad (2.6)$$

$v_{\{\lambda\}, \{\mu\}}(\beta)$ is a coefficient. For $\beta = 1$ the Jack symmetric function coincides with the Schur functions for free fermions. The norm of the Jack symmetric functions is defined by

$$\langle J_{\{\lambda\}}^\beta, J_{\{\lambda\}}^\beta \rangle = j_{\{\lambda\}}^{(\beta)}. \quad (2.7)$$

In particular, $j_{\{0\}} = 1$.

The Jack symmetric functions have a very interesting property of *duality* $\beta \longleftrightarrow 1/\beta$. Let ω be an automorphism on the ring of symmetric polynomials, defined by

$$\omega(p_n) = -\frac{(-1)^n p_n}{\beta}. \quad (2.8)$$

Then duality transformation transforms a Jack polynomial into its dual:

$$\omega J_{\{\lambda\}}^\beta = j_{\{\lambda\}}^{(\beta)} J_{\{\lambda'\}}^{(1/\beta)} \quad (2.9)$$

Since power sums form a basis of the ring of symmetric functions, Jack symmetric functions can be written as a functional of power sums p_n ;

$$J_{\{\lambda\}}^\beta(x_1, x_2, \dots, x_N) = J_{\{\lambda\}}^\beta(\{p_n\}). \quad (2.10)$$

With this notation, the above duality tells us that

$$J_{\{\lambda\}}^{(\beta)}(\{-\frac{p_n}{\beta}\}) = (-1)^{|\lambda|} j_{\{\lambda\}}^{(\beta)} J_{\{\lambda'\}}^{(1/\beta)}(\{p_n\}). \quad (2.11)$$

Two dual Jack symmetric functions are related by the following duality relation:

$$\prod_{i=1}^N \prod_{j=1}^M (1 - x_i y_j) = \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda\}}^{(\beta)}(x_i) J_{\{\lambda'\}}^{(1/\beta)}(y_j) \quad (2.12)$$

Here $\{\lambda'\}$ is a conjugate partition to $\{\lambda\}$ and partitions $\{\lambda\}$ are summed over those that satisfy $l(\lambda) \leq N$ and $l(\lambda') \leq M$.

2.2 Calogero-Sutherland Model

Jack symmetric functions introduced above give eigenfunctions of the Calogero-Sutherland (CS) model, an exactly solvable model in 1d with inverse-square interactions on a circle. The Hamiltonian is given by (1.1). Using z variables instead of x , defined by $z = e^{i2\pi x/L}$, the Hamiltonian is rewritten as

$$H_{CS} = \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \left\{ \sum (z_i \partial_i)^2 - 2\beta(\beta - 1) \sum_{i < j} \frac{z_i z_j}{(z_i - z_j)^2} \right\} \quad (2.13)$$

where $\partial_i = \partial/\partial z_i$. The groundstate has the Jastrow form:

$$\psi_0(z_i) = \prod_{i < j} (z_i - z_j)^\beta \prod_i z_i^{-\beta N/2}. \quad (2.14)$$

(N is a particle number.) Wave function of an excited state is written as $\psi(z_i) = f(z_i) \psi_0(z_i)$ where $f(z_i)$ is a symmetric polynomial. The CS Hamiltonian (2.13) acts on $f(z_i)$ as

$$\begin{aligned} H_{CS} \psi(z_i) &= (\tilde{H}_{CS} f(z_i)) \psi_0, \quad \tilde{H}_{CS} = \tilde{H} + \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \frac{N^3 - N}{12} \beta^2 \\ \tilde{H} &= \frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \left\{ \sum (z_i \partial_i)^2 + \beta \sum_{i < j} \frac{z_i + z_j}{z_i - z_j} (z_i \partial_i - z_j \partial_j) \right\}. \end{aligned} \quad (2.15)$$

Its eigenfunction is characterized by a Young tableau $\{\lambda\}$ [19]:

$$f(z_i) = J_{\{\lambda\}}^{(\beta)}(z_i) \prod_i (z_i)^p \quad (2.16)$$

whose eigenvalue of \tilde{H}_{CS} is

$$\begin{aligned} &\frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \sum_i^N (m_i)^2 \\ m_i &= p + \lambda_i + \beta \left(\frac{N+1}{2} - i \right). \end{aligned} \quad (2.17)$$

When we write $p = q - \beta N/2$,

$$m_i = q + \lambda_i - \frac{2i-1}{2} \beta. \quad (2.18)$$

The above state is also an eigenstate of the momentum operator

$$\hat{P} = \frac{2\pi}{L} \sum z_i \partial_i \quad (2.19)$$

with an eigenvalue $\sum_i^N m_i$. In later sections, this momentum operator is identified with the Virasoro generator while the CS Hamiltonian is identified with a higher-spin (spin 3) conserved charge.

2.3 Collective Hamiltonian

The CS Hamiltonian \tilde{H} acts on a space of symmetric functions and since the ring of symmetric functions are generated by power sums $p_n = \sum z_i^n$, we can rewrite the CS Hamiltonian in terms of creation and annihilation of power sums. By using the standard technique of collective field theory [20, 21, 22], we can obtain the collective Hamiltonian of the CS model. f can be expanded in terms of the power sums as

$$f = \sum_k f\{n_1, \dots, n_k\} \prod_{i=1}^k p_{n_i} \equiv f\{n_1, \dots, n_k\} |n_1, \dots, n_k\rangle \quad (2.20)$$

where $f\{n_i\}$ are expansion coefficients. Annihilation and creation operators of power sums ($n > 0$) are defined by $|\{n\}\rangle \equiv a_n^+ |\{0\}\rangle$ and $a_n |\{n\}\rangle = n |\{0\}\rangle$. They satisfy the commutation relation $[a_n, a_m^+] = n \delta_{m,n}$ and the "vacuum" conditions $a_n |\{0\}\rangle = 0$. Note that they are not hermite conjugate in general: $(a_n)^\dagger \neq a_n^+$. In terms of a_n and a_n^+ , we get the collective field theory Hamiltonian for the CS Hamiltonian \tilde{H} :

$$\tilde{H}_{coll} = \sum_{n>0} (1 - \beta) n a_n^+ a_n + N \beta \sum_{n>0} a_n^+ a_n + \sum_{n, n' > 0} (\beta a_n^+ a_{n'}^+ a_{n+n'} + a_{n+n'}^+ a_n a_{n'}). \quad (2.21)$$

The eigenstates are given by

$$J_{\{\lambda\}}^{(\beta)}(\{a_n^+\}) |\{0\}\rangle. \quad (2.22)$$

The second term in the Hamiltonian contains the particle number, but as we will see later, this N dependence can be absorbed in the last term.

3 $c = 1$ CFT

3.1 Vertex Operators

In this section we consider a compactified free boson with radius $r = \sqrt{\beta}$. We only consider the holomorphic part here. β can take any positive number but when we consider the statistics of vertex operators or the structure of pseudo-Dirac sea, we restrict it to an integer. It is straightforward to generalize it to a rational number $\beta = p/q$.

First the bosonic field is expanded as

$$\phi(z) = \hat{q} - i\alpha_0 \log z + i \sum_{n \neq 0} \frac{\alpha_n z^{-n}}{n}. \quad (3.1)$$

Commutation relations are $[\hat{q}, \alpha_0] = 1$ and $[\alpha_n, \alpha_{n'}^\dagger] = n\delta_{n,n'}$. The bosonic field satisfies the operator product expansion (OPE):

$$\phi(z)\phi(z') \sim -\log(z - z'). \quad (3.2)$$

Energy-momentum tensor is defined by $T(z) \equiv -(1/2) : (\partial\phi(z))^2 :$ and we define the charge by the zero mode of the current operator $J(z) \equiv i\sqrt{\beta}\partial\phi(z)$, that is, $\sqrt{\beta}\alpha_0$. The Fock space of this compactified boson is characterized by the charge q and bosonic excitations constructed by α_n for $n < 0$. The charge must take integer values.

We can now define two kinds of vertex operators. One is anyon vertex operators and the other is bosonic or fermionic vertex operators corresponding to even β or odd β . Anyon vertex operators are defined by

$$\Phi^\pm(z) =: e^{\pm i\phi(z)/\sqrt{\beta}} :. \quad (3.3)$$

The scaling dimension of the operators is $1/(2\beta)$ and they have anyonic statistics $\theta = \pm\pi/\beta$. They carry charge ± 1 . OPE's are given by

$$\begin{aligned} \Phi^\pm(z)\Phi^\pm(z') &= (z - z')^{1/\beta} : e^{\pm i(\phi(z) + \phi(z'))/\sqrt{\beta}} : \sim (z - z')^{1/\beta} \\ \Phi^+(z)\Phi^-(z') &= (z - z')^{-1/\beta} : e^{i(\phi(z) - \phi(z'))/\beta} : \\ &\sim (z - z')^{-1/\beta} (1 + (z - z') \frac{i}{\sqrt{\beta}} \partial\phi(z') + \dots). \end{aligned} \quad (3.4)$$

Mode expansion on a charge q sector is given by

$$\Phi^\pm(z)|q\rangle = \sum_{n \in \mathbf{Z}} \Phi_{-n \mp \frac{q}{\beta} - \frac{1}{2\beta}}^\pm z^{n \pm \frac{q}{\beta}} |q\rangle \quad (3.5)$$

By following the standard procedure [23] we can derive the following generalized commutation relations³ :

$$\begin{aligned} \sum_{l=0}^{\infty} C_l^{1-1/\beta} (\Phi_{1-n_1 \mp \frac{q \pm 1}{\beta} - l - \frac{1}{2\beta}}^\pm \Phi_{-n_2 \mp \frac{q}{\beta} + l - \frac{1}{2\beta}}^\pm + (n_1 \longleftrightarrow n_2)) |q\rangle &= 0 \\ \sum_{l=0}^{\infty} C_l^{-1+1/\beta} (\Phi_{-1-n_1 - \frac{q-1}{\beta} - l - \frac{1}{2\beta}}^+ \Phi_{-n_2 + \frac{q}{\beta} + l - \frac{1}{2\beta}}^- \\ + \Phi_{-1-n_2 + \frac{q+1}{\beta} - l - \frac{1}{2\beta}}^- \Phi_{-n_1 - \frac{q}{\beta} + l - \frac{1}{2\beta}}^+) |q\rangle &= \delta_{n_1+n_2, -1} |q\rangle. \end{aligned} \quad (3.6)$$

The coefficients are defined by $(1-x)^\alpha = \sum_{l=0}^{\infty} C_l^\alpha x^l$. When $\beta = 1$, these commutation relations reduce to the usual fermion anti-commutation relations.

The other kind of vertex operators are

$$\Psi^\pm(z) =: e^{\pm i \sqrt{\beta} \phi(z)} :. \quad (3.7)$$

They have scaling dimension $\beta/2$ and statistics $\pm\pi\beta$. Let's call this kind of vertex operator *super fermion* vertex operator since, if β is an integer, it has an integer-valued (times π) exchange phase as the usual free bosons or free fermions. They carry charge $\pm\beta$. OPE's are given by

$$\begin{aligned} \Psi^\pm(z) \Psi^\pm(z') &= (z-z')^\beta : e^{\pm i \sqrt{\beta}(\phi(z)+\phi(z'))} : \sim (z-z')^\beta \\ \Psi^+(z) \Psi^-(z') &= (z-z')^{-\beta} : e^{i \sqrt{\beta}(\phi(z)-\phi(z'))} : \\ &\sim (z-z')^{-\beta} (1 + (z-z') i \sqrt{\beta} \partial \phi(z') + \dots). \end{aligned} \quad (3.8)$$

Mode expansions on a charge q sector are similarly given by

$$\Psi^\pm(z)|q\rangle = \sum_{n \in \mathbf{Z}} \Psi_{-n \mp q - \frac{\beta}{2}}^\pm z^{n \pm q} |q\rangle. \quad (3.9)$$

³They are derived by [14] for $\beta = 2$ and used to obtain a fermionic representation of Virasoro characters and $SU(2)$ level 1 Kac-Moody algebra.

In this case, the charge q can be absorbed by shift of n . Generalized commutation relations for Ψ and Ψ or Φ and Ψ are similarly given. Here we only comment that there is a simple relation for Ψ 's:

$$\Psi_{-n_1-\frac{\beta}{2}}^{\pm} \Psi_{-n_2-\frac{\beta}{2}}^{\pm} - (-1)^{\beta} \Psi_{-n_2-\frac{\beta}{2}}^{\pm} \Psi_{-n_1-\frac{\beta}{2}}^{\pm} = 0 \quad (3.10)$$

3.2 Pseudo-Dirac Sea — Vacuum

By using the above mode expansion, we can define the pseudo-Dirac sea with an infinite set of pseudo-momenta. First it is easy to check that the vacuum state for the boson annihilation operators with charge q satisfies the following conditions:

$$\begin{aligned} \Phi_{n \mp \frac{q}{\beta} - \frac{1}{2\beta}}^{\pm} |q\rangle &= 0, \quad n \geq 1 \\ \Psi_{n \mp q - \frac{\beta}{2}}^{\pm} |q\rangle &= 0, \quad n \geq 1. \end{aligned} \quad (3.11)$$

Similar to the free fermion case eq.(1.3), we can express charge 0 vacuum in terms of an infinite set of pseudo-momenta as follows:

$$|0\rangle = \Psi_{\frac{\beta}{2}}^{\dagger} |-\beta\rangle = \Psi_{\frac{\beta}{2}}^{\dagger} \Psi_{\frac{3\beta}{2}}^{\dagger} \cdots |-\infty\rangle = \prod_{i=0}^{\infty} \Psi_{-m_i^0}^+ |-\infty\rangle \quad (3.12)$$

where $m_i^0 = -(2i-1)\beta/2$. Different from the free fermion case, however, commutation relations between Ψ^+ and Ψ^- are complicated and such states as $\prod_{i=0}^{\infty} \Psi_{-m_i}^+ |-\infty\rangle$ are not orthogonal to each other in general. We must therefore find more general framework to construct pseudo-momentum representation or pseudo-Dirac sea. As we can see later, there are two trivial cases (other than vacuum) that the above way of construction of states works. One is a state that can be created by a single anyonic vertex operator Φ^+ and the other is a state that can be created by a single super-fermionic vertex operator Ψ^+ . In the language of Young tableau, these states correspond to tableau that has only single row or single column.

3.3 Multi Anyon States

In order to obtain a general framework for pseudo-Dirac sea, we first consider multi anyon states. The duality relation (2.12) plays an important role. The

left hand side of the equation can be rewritten as

$$\begin{aligned}\prod(1 - x_i y_j) &= \exp(\log \prod(1 - x_i y_j)) = \exp[-\sum_{n=1}^{\infty} \sum_{i,j} \frac{(x_i y_j)^n}{n}] \\ &= \exp[-\sum_{n=1}^{\infty} \sum_{j=1}^M \frac{p_n(x)(y_j)^n}{n}].\end{aligned}\quad (3.13)$$

Here $p_n(x) = \sum_i^N (x_i)^n$. Therefore, the duality relation (2.12) becomes

$$\exp[-\sum_{n=1}^{\infty} \sum_{j=1}^M \frac{p_n(x)(y_j)^n}{n}] = \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda\}}^{(\beta)}(\{p_n\}) J_{\{\lambda'\}}^{(1/\beta)}(y_j). \quad (3.14)$$

The sums are restricted to $l(\lambda) \leq N$ and $l(\lambda') \leq M$. If we put formally $N \rightarrow \infty$ and $p_n(x) = \mp \alpha_{-n}/\sqrt{\beta}$, we get

$$\exp[\pm \sum_{n=1}^{\infty} \sum_{j=1}^M \frac{\alpha_{-n}(y_j)^n}{\sqrt{\beta} n}] = \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda'\}}^{(1/\beta)}(y_j) J_{\{\lambda\}}^{(\beta)}(\{\mp \frac{\alpha_{-n}}{\sqrt{\beta}}\}). \quad (3.15)$$

Sums are restricted to $l(\lambda') \leq M$.

M -anyon states on the vacuum with charge \tilde{q} are obtained by multiplying M anyon vertex operators:

$$\begin{aligned}\prod_{j=1}^M \Phi^{\pm}(z_j) |\tilde{q}\rangle &= \prod_{i < j} (z_i - z_j)^{1/\beta} : e^{\pm i \sum \phi(z_j)/\sqrt{\beta}} : |\tilde{q}\rangle \\ &= \prod_{i < j} (z_i - z_j)^{1/\beta} \prod_j^M (z_j)^{\pm \tilde{q}/\beta} \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda'\}}^{(1/\beta)}(z_j) J_{\{\lambda\}}^{(\beta)}(\{\mp \frac{\alpha_{-n}}{\sqrt{\beta}}\}) | \pm M + \tilde{q} \rangle \\ &= \prod_{i < j} (z_i - z_j)^{1/\beta} \prod_j^M (z_j)^{\pm \tilde{q}/\beta} \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda'\}}^{(1/\beta)}(z_j) (j_{\lambda}^{\beta})^{1/2} |\{\lambda\}, \pm M + \tilde{q}\rangle_{\mp}.\end{aligned}\quad (3.16)$$

Here we defined

$$|\{\lambda\}, q\rangle_{\mp} \equiv (j_{\lambda}^{\beta})^{-1/2} J_{\{\lambda\}}^{(\beta)}(\{\mp \frac{\alpha_{-n}}{\sqrt{\beta}}\}) |q\rangle. \quad (3.17)$$

At $\beta = 2$ this state was identified with the motif-represented state of $SU(2)$ level 1 Kac-Moody algebra [13]. These states are shown to be orthogonal

to each other and normalized to unity. This can be proved as follows: If we define *power-sum* states corresponding to a partition $\{\lambda\}$, created by the bosonic creation operators $\mp \alpha_{-n} \sqrt{\beta}$,

$$|\alpha_{\{\lambda\}}, q\rangle_{\mp} \equiv \prod_{i=1}^{l(\lambda)} \left(\mp \frac{\alpha_{\lambda_i}^{\dagger}}{\sqrt{\beta}} \right) |q\rangle, \quad (3.18)$$

it is easy to prove that these states satisfy

$${}_{\mp}\langle \alpha_{\{\lambda\}}, q | \alpha_{\{\mu\}}, q \rangle_{\mp} = \delta_{\{\lambda\}, \{\mu\}} \beta^{-l(\lambda)} z_{\lambda}. \quad (3.19)$$

Therefore, from the definition of the Jack symmetric functions, those states $|\{\lambda\}, q\rangle_{\mp}$ defined above are orthogonal to each other and normalized to unity. Moreover, due to the completeness of Jack symmetric functions, they form a complete basis in the $c = 1$ CFT if we consider all partitions and all integer charges.

From the duality transformation eq.(2.11), these states can be written in another form:

$$|\{\lambda\}, q\rangle_{\mp} = (-1)^{|\lambda|} (j_{\lambda}^{\beta})^{1/2} J_{\{\lambda'\}}^{(1/\beta)}(\{\pm \sqrt{\beta} \alpha_{-n}\}) |q\rangle. \quad (3.20)$$

For a partition $\{\lambda\}$ where $l(\lambda') \leq M$, from eq.(3.16) and orthonormal property of $|\{\lambda\}, q\rangle_{\mp}$, we get

$${}_{\mp}\langle \{\lambda\}, \pm M + \tilde{q} | \prod_{j=1}^M \Phi^{\pm}(z_j) | \tilde{q} \rangle = \prod_{i < j} (z_i - z_j)^{1/\beta} \prod_j^M (z_j)^{\pm \tilde{q}/\beta} (-1)^{|\lambda|} J_{\{\lambda'\}}^{(1/\beta)}(z_j) (j_{\lambda}^{\beta})^{1/2} \quad (3.21)$$

The r.h.s is the eigenfunction of the CS Hamiltonian with coupling constant $1/\beta$. This shows that the eigenfunction of $1/\beta$ CS model gives "bound state" wave functions for M anyon states.

3.4 Super Fermion Picture

Next let's consider states created by *super fermion* vertex operators Ψ^{\pm} . Similar to the multi anyon case, action of multi super-fermion vertex operators can be evaluated as follows. First note that eq.(3.14) can be also written in the following form:

$$\exp\left[-\sum_{n=1}^{\infty} \sum_{i=1}^N \frac{p_n(y)(x_i)^n}{n}\right] = \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda\}}^{(\beta)}(x_i) J_{\{\lambda'\}}^{(1/\beta)}(\{p_n\}). \quad (3.22)$$

Then put formally $M \rightarrow \infty$ and $p_n(y) = \mp \sqrt{\beta} \alpha_{-n}$ and we get

$$\exp[\pm \sum_{n=1}^{\infty} \sum_{i=1}^N \sqrt{\beta} \frac{\alpha_{-n}(x_i)^n}{n}] = \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda\}}^{(\beta)}(x_i) J_{\{\lambda'\}}^{(1/\beta)}(\{\mp \sqrt{\beta} \alpha_{-n}\}). \quad (3.23)$$

Sums are restricted to $l(\lambda) \leq N$. Using this equation and eq.(3.20), action of N super-fermion vertex operators on vacuum is obtained:

$$\begin{aligned} \prod_{i=1}^N \Psi^{\pm}(z_i) |\tilde{q}\rangle &= \prod_{i < j} (z_i - z_j)^{\beta} : e^{\pm i \sqrt{\beta} \sum \phi(z_j)} : |\tilde{q}\rangle \\ &= \prod_{i < j} (z_i - z_j)^{\beta} \prod_i (z_i)^{\pm \tilde{q}} \sum_{\{\lambda\}} (-1)^{|\lambda|} J_{\{\lambda\}}^{(\beta)}(z_i) J_{\{\lambda'\}}^{(1/\beta)}(\{\mp \sqrt{\beta} \alpha_{-n}\}) |\pm N\beta + \tilde{q}\rangle \\ &= \prod_{i < j} (z_i - z_j)^{\beta} \prod_i (z_i)^{\pm \tilde{q}} \sum_{\{\lambda\}} J_{\{\lambda\}}^{(\beta)}(z_i) (j_{\lambda}^{\beta})^{-1/2} |\{\lambda\}, \pm N\beta + \tilde{q}\rangle_{\pm}. \end{aligned} \quad (3.24)$$

Summations are over partitions $l(\lambda) \leq N$. Therefore, for a partition $l(\lambda) \leq N$, bound state wave functions of multi super-fermion states are given by

$${}_{\pm} \langle \{\lambda\}, \pm q | \prod_{i=1}^N \Psi^{\pm}(z_i) | \pm(q - N\beta) \rangle = \prod_{i < j} (z_i - z_j)^{\beta} \prod_i (z_i)^{q - N\beta} J_{\{\lambda\}}^{(\beta)}(z_1, \dots, z_N) (j_{\lambda}^{\beta})^{-1/2} \quad (3.25)$$

Here we redefined the charge so that the bra-state has charge $\pm q$. The r.h.s. is the eigenfunctions of the CS Hamiltonian with coupling constant β . This shows that the super-fermion vertex operator can be interpreted as a field operator of interacting fermions of CS model. The $c = 1$ CFT at $r = \sqrt{\beta}$, therefore, gives a second quantized formulation of CS model with coupling β . Furthermore, eq.(3.21) and eq.(3.25) show that there are dual picture of $c = 1$ CFT, "anyon" picture and "super-fermion" picture. In the anyon picture, multi-anyon bound states are characterized by $1/\beta$ CS eigenfunctions. On the other hand, in the super fermionic picture, multi fermion bound states are characterized by β CS eigenfunctions.

3.5 Pseudo-Momentum Representation

Now we get the framework to study the pseudo-Dirac sea structure of the $c = 1$ CFT. The r.h.s. of eq.(3.25) is an eigenfunction of the CS Hamiltonian

H_{CS} with an eigenvalue

$$\frac{1}{2} \left(\frac{2\pi}{L} \right)^2 \sum_i^N (m_i)^2 \quad (3.26)$$

where m_i is given by

$$m_i = q + \lambda_i - \beta \frac{2i-1}{2}. \quad (3.27)$$

If we put formally $N \rightarrow \infty$, it is characterized by an infinite set of pseudo-momenta. Since the state $|\{\lambda\}, \pm q\rangle_{\pm}$ is independent of N , we can label the state by this infinite set of pseudo-momenta. For the vacuum state with charge 0, this infinite set $\{m_i^0\}$ coincides with (1.3). Since $\lambda_1 > \lambda_2 > \dots$, the pseudo-momenta ($m_i > m_j$ if $i < j$) can take any integer value with the only constraints $|m_i - m_j| \geq \beta$. From now on, we write

$$|\{m_i\}\rangle_{\pm} \equiv |\{\lambda\}, \pm q\rangle_{\pm} \quad (3.28)$$

Each (\pm) pseudo-momenta representation of states form an orthonormal and complete basis in the $c = 1$ CFT.

Here let's see how the Virasoro generator acts on the pseudo-momentum representation. From the operator product expansion of the Virasoro generator $T(z) \equiv -(1/2) : (\partial\phi(z))^2 :$ and the vertex operator Ψ , we get the following equation for an analytic function $f(z)$:

$$\begin{aligned} \langle \{m_i\} | \oint \frac{dz}{2\pi i} f(z) T(z) \prod \Psi^+(z_i) | \tilde{p} \rangle = \\ \oint \frac{dz}{2\pi i} f(z) \sum_i \left(\frac{\beta/2}{(z-z_i)^2} + \frac{1}{z-z_i} \partial_i \right) \langle \{m_i\} | \prod \Psi^+(z_i) | \tilde{p} \rangle. \end{aligned} \quad (3.29)$$

Virasoro generators L_n are given by putting $f(z) = z^{n+1}$. If we put $f(z) = z$ and $N \rightarrow \infty$, we get

$$L_0 |\{m_i\}\rangle_{\pm} = \left(\sum_i^{\infty} m_i \right) |\{m_i\}\rangle_{\pm}. \quad (3.30)$$

On the other hand the action of L_0 is also given by eq.(3.28) and eq.(3.17) and the eigenvalue of L_0 is

$$\sum \lambda_i + \frac{q^2}{2\beta}. \quad (3.31)$$

Regularizing the infinite sum of m_i in eq.(3.30), it coincides with this eigenvalue. The second term is the Casimir energy of the charge q vacuum.

3.6 Collective Hamiltonian

In this subsection we give a collective Hamiltonian acting on the state $|\{m_i\}\rangle_{\pm}$ that gives an eigenvalue $\sum_i^{\infty} (m_i)^2$. From eq.(3.28), eq.(3.17) and eq.(2.22), such collective Hamiltonian for states $|\{m_i\}\rangle_{\pm}$ is obtained by replacing a_n^+ in (2.21) by $\mp\sqrt{\beta}\alpha_n^{\dagger}$. From commutation relations, a_n must be replaced by $\mp\alpha_n\sqrt{\beta}$. Then we get the collective Hamiltonian

$$\tilde{H}_{coll}^{\beta,\pm} = \sum_{n>0} (1-\beta)n\alpha_n^{\dagger}\alpha_n + N\beta \sum_{n>0} \alpha_n^{\dagger}\alpha_n \mp \sqrt{\beta} \sum_{n,n'>0} (\alpha_n^{\dagger}\alpha_{n'}^{\dagger}\alpha_{n+n'} + \alpha_{n+n'}^{\dagger}\alpha_n\alpha_{n'}). \quad (3.32)$$

Comparing with eq.(2.14) and eq.(3.25), this collective Hamiltonian should be understood as acting on a state with charge $\pm q$ that satisfies $q - N\beta = -N\beta/2$. Hence, for a general charged state, the collective Hamiltonian is obtained by replacing N by α_0 through the relation $\alpha_0\sqrt{\beta} = \pm q = \pm N\beta/2$:

$$\tilde{H}_{coll}^{\beta,\pm} = \sum_{n>0} (1-\beta)n\alpha_n^{\dagger}\alpha_n \mp \sqrt{\beta} \sum_{n,n'\geq 0} (\alpha_n^{\dagger}\alpha_{n'}^{\dagger}\alpha_{n+n'} + \alpha_{n+n'}^{\dagger}\alpha_n\alpha_{n'}). \quad (3.33)$$

Of course, this Hamiltonian is hermite under the usual measure for the free boson Fock space. Eigenvalue of $\tilde{H}_{coll}^{\beta,\pm}$ on $|\{m_i\}\rangle_{\pm}$ is

$$\tilde{H}_{coll}^{\beta,\pm} |\{m_i\}\rangle_{\pm} = \sum_i^{\infty} ((m_i)^2 - (m_i^0)^2) |\{m_i\}\rangle_{\pm}. \quad (3.34)$$

The second term in the Hamiltonian is written in a local form $\frac{1}{3} \oint \frac{dz}{2\pi i} : (\partial\phi(z))^3 : z^2$ but the first term cannot. If we write the creation (annihilation) part of the bosonic field by ϕ_{\pm} , the collective Hamiltonian can be written as

$$\tilde{H}_{coll}^{\beta,\pm} = \oint \frac{dz}{2\pi i} \left(\mp \frac{\sqrt{\beta}}{3} : (\partial\phi(z))^3 : z^2 + (1-\beta) : \partial\phi_+(z)(z\partial)^2\phi_-(z) : \right). \quad (3.35)$$

This collective Hamiltonian for the CS model is spin 3 conserved charge. This indicates that there are infinitely many conserved charges $\sum_i^{\infty} ((m_i)^p - (m_i^0)^p)$ in this $c = 1$ CFT. Indeed it has been known that the Calogero-Sutherland model is exactly solvable and has an infinitely many conserved charges. In the paper [24], these higher conserved charges are discussed. Different from the free fermion case, these charges will be non-local in general. As the infinite charges in the free fermion obey $c = 1$ $W_{1+\infty}$ algebra, they

will also obey some *deformed* $c = 1$ $W_{1+\infty}$ algebra. It will be interesting to find the full algebra and its representation theory.

Eq.(3.21) shows that there is another Hamiltonian that acts on the pseudo-momentum states, CS Hamiltonian with coupling constant $1/\beta$. Its collective Hamiltonian is given by replacing β by $1/\beta$ in $\tilde{H}_{coll}^{\beta,\pm}$. This becomes

$$\tilde{H}_{coll}^{1/\beta,\pm} = \frac{-1}{\beta} \tilde{H}_{coll}^{\beta,\mp} \quad (3.36)$$

and gives the same collective Hamiltonian as that of β . In other words, the same conserved charge in the $c = 1$ CFT gives different representation, CS Hamiltonian with coupling β for multi super-fermion states and CS Hamiltonian with coupling $1/\beta$ for multi anyon states. This duality will be discussed in detail in a separate paper.

4 Conclusion and Discussions

In this paper we study the $c = 1$ conformal field theory with respect to the multi anyon and “super-fermion” states. By using the technique of Calogero-Sutherland model, we show that at radius (of a compactified boson) $r = \sqrt{\beta}$, where β is an integer, the Fock space is labeled by an infinite set of pseudo-momenta. This gives a natural generalization of boson-fermion correspondence to boson-anyon correspondence in (1+1) dimensions. Bouwknegt et.al. [14] recently obtained the Virasoro character at $r = \sqrt{2}$ point by using spinon description and Yangian symmetry. It is an open problem to obtain the character at other radius in this anyon basis.

This anyon picture will be generalized to other conformal field theories. For example, $c < 1$ Virasoro minimal models can be obtained from $c = 1$ CFT by reducing the Fock space through Feigin-Fuchs construction. The anyon basis of $c = 1$ CFT will give a powerful tool to study the minimal models. This direction is discussed by a paper [25].

Dynamical correlation functions for the Calogero-Sutherland model have been recently obtained [26, 22]. Our formulation of the Calogero-Sutherland model in terms of $c = 1$ CFT will give a second quantized formulation of the dynamical correlations. It will be published elsewhere.

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